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Measurement description by means of a nonlinear Schrödinger equation

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Abstract. A new general formulation for nonlinear quantum mechanics is discussed. It is shown that a nonlinear deterministic evolution with random variables can model a measurement as a dynamical process. Phases of state vectors are treated as random variables of the theory.

1. Introduction

The quantum measurement problem consists in conciliation of the way in which probabilities are calculated with deterministic and linear Schrödinger evolution. There are several approaches to solving the problem. The first approach denies the necessity of describing measurements on the basis of suitable interpretation [1, 2]. The second attributes the 'collapse of a wavefunction' to an inevitable interaction with an environment [3]. The third consists in replacement of the Schrödinger equation by another one capable of describing measurements.

We discuss the last opportunity. This choice is based on the fact that standard quantum mechanics is a non-relativistic theory. The notion of the wavefunction is valid for a limited region of spacetime and small velocities whereas an act of observation requires processes of particle creation and annihilation, which belong to the domain of relativity. Though relativistic theory is considered to be linear, the transition to the non-relativistic case may produce nonlinear corrections to the Schrödinger equation, which revealed themselves in the measurement process.

Various models of nonlinear quantum mechanics have been discussed from the beginning of quantum theory up until now [4–14]. Recently progress has been made in understanding what kind of nonlinearities can be used to describe state vector reduction as a dynamical process [15–22]. We consider the following class of nonlinear equations conserving the norm of a state vector $|\psi\rangle$ ($\hbar = 1$):

$$\mathrm{i}\mathrm{d}|\psi\rangle/\mathrm{d}t = \mathcal{H}|\psi\rangle = H|\psi\rangle + (1 - P_\psi)U|\psi\rangle \tag{1}$$

where H is the Hermitian part of the Hamiltonian $\mathcal{H}$, $(1 - P_\psi)U$ is the non-Hermitian term, and $P_\psi = |\psi\rangle\langle\psi|$ is the projection operator. It is necessary to note that U can be an arbitrary linear or nonlinear operator.

Various generalizations of the Schrödinger equation can be found from (1) with various choices of H and U. For example, the Bohm–Bub theory [5] corresponds to the choice $H = 0$,

$$U = \frac{\mathrm{i}\gamma}{2}\sum_n \frac{|\langle\varphi_n|\psi\rangle|^2}{|\zeta_n|^2}|\varphi_n\rangle\langle\varphi_n|$$

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where γ is a coupling constant, ζ_n are complex constants, and from here on $|\varphi_n\rangle$ belongs to a complete set of orthonormal vectors. The Pearle theory of 1976 [8] can be found with

$$H = \langle\psi|A|\psi\rangle \qquad \text{and} \qquad U = \sum_n \frac{\langle\psi|A|\psi_n\rangle}{\langle\psi|\varphi_n\rangle}|\varphi_n\rangle\langle\varphi_n|$$

where A is an arbitrary linear self-adjoint operator. The Gisin theory of 1981 [10] is given by $H = U/k\mathrm{i} = H_0$, where H_0 is the linear Hamiltonian and k is a real constant. The particular case of the Gisin theory of 1989 [17] as well as the Ghirardi *et al* theory of 1989 [18] corresponds to

$$H = H_0 \qquad U = \mathrm{i}\boldsymbol{A}\,\mathrm{d}\boldsymbol{W} - \tfrac{1}{2}\gamma(\boldsymbol{A} - \langle\psi|\boldsymbol{A}|\psi\rangle)^2$$

where $\boldsymbol{A} = \{A_i\}$ is a set of self-adjoint operators and $\boldsymbol{W} = \{W_i\}$ is a real Wiener process with $\langle \mathrm{d}W_i\,\mathrm{d}W_j\rangle = \gamma\delta_{ij}\,\mathrm{d}t$. The choice of the Hamiltonian matrix H in the form $H_{ik} = \partial^2 w/(\partial\psi_i^*\partial\psi_k)$ and $U = 0$ leads to the Weinberg theory [20]. The generalization by Dodonov and Mizrahi [14] of the Doebner and Goldin equation [13] can be found with

$$H = 0 \qquad \text{and} \qquad U = \mathrm{i}\left[\sum_n \frac{\langle\psi|Q^+|\varphi_n\rangle\langle\varphi_n|Q|\psi\rangle}{\langle\psi|\varphi_n\rangle\langle\varphi_n|\psi\rangle}|\varphi_n\rangle\langle\varphi_n| - Q^+Q\right]$$

or, equivalently, with

$$H = 0 \qquad \text{and} \qquad U = \mathrm{i}\left[\sum_n \frac{\langle\psi|Q^+|\varphi_n\rangle}{\langle\psi|\varphi_n\rangle}|\varphi_n\rangle\langle\varphi_n| - Q^+\right]Q$$

where Q *is a linear operator.*

It is evident from the above examples that (1) provides a suitable basis for nonlinear generalizations of quantum mechanics. It was discovered by Gisin [10,23] without the emphasis on the possibility of U being a nonlinear operator. Then, it was independently found in [21,24].

Equation (1) has several important features. First, different Hamiltonians can result in identical motion equations. This is the case, e.g., for Hamiltonians $\mathcal{H}$ and $\mathcal{H} + A(1 - P_\psi)$, where A is an arbitrary operator. This means that the division of the nonlinear Hamiltonian into Hermitian and non-Hermitian parts is arbitrary in a class of equivalent Hamiltonians. We can point out the completely Hermitian Hamiltonian $\mathcal{H} = H + (1 - P_\psi)U + U^+(1 - P_\psi)$ which also results in motion equation (1). Conversely, any norm-conserving nonlinear Hamiltonian has an equivalent one presented by (1):

$$\mathcal{H} = (\mathcal{H} + \mathcal{H}^+)/2 + (1 - P_\psi)(\mathcal{H} - \mathcal{H}^+)/2.$$

Thus, the Hamiltonian action on a state vector instead of the Hamiltonian itself is of interest in a nonlinear theory.

Second, a U operator may have a trivial part connected with the rescaling of a global phase of a state vector. Any U operator can be written in the form $U = R + Q$ with $R^+ = R$ and $Q^+ = -Q$. Then, the Hermitian part R can be moved to the Hermitian part of the Hamiltonian by a change of the phase of the state vector. Namely, the state vector

$$|\tilde{\psi}\rangle = \exp(-\mathrm{i}\chi)|\psi\rangle \qquad \text{with} \qquad \chi = \int_0^t \langle\psi|R|\psi\rangle\,\mathrm{d}t$$

obeys (1) with

$$\bar{H} = H(\exp(i\chi)|\bar{\psi}\rangle) + R(\exp(i\chi)|\bar{\psi}\rangle) \qquad \text{and} \qquad \bar{U} = Q(\exp(i\chi)|\bar{\psi}\rangle).$$

Therefore, we can impose the condition $U^+ = -U$ on the nonlinear operator without loss of generality.

The most remarkable feature of (1) is the projection property exhibited by some nonlinear Hamiltonians. As an example, let us take the Hamiltonian $\mathcal{H}$ in the form

$$H = 0 \qquad \text{and} \qquad U = \sum_n i\gamma q_n |\varphi_n\rangle\langle\varphi_n| \tag{2}$$

where q_n are complex constants and γ is a positive coupling constant [21]. Then, the dynamics (1) causes almost each initial state of the system to evolve to the state with the maximum value of $\mathrm{Re}(q_n)$. This evolution is completely strange to linear quantum mechanics and allows us to construct simple schemes for measurement modelling. Here, a measurement description implies deterministic dynamics which specifies different experimental outcomes by means of some random variables or initial conditions (say, a phase of an initial state vector). Naturally, the probability of finding the system in a given final state must agree with experimental observations.

2. The description of measurements

We model only non-destructive measurements consisting of the interaction of a quantum system with a classical detector. We assume that the system dynamics in the measurement process is described by equation (1), which is closed in terms of a system state vector. The incorporation of a state vector of the detector creates no problem. A classical detector, which is represented by a nonlinear U operator, determines the 'strength' of the nonlinear terms and a set of state vectors observed after measurement. The important question as to how such a nonlinear potential is produced by a system interaction with individual atoms of the detector is not clear and will not be discussed here. The final state vectors of the system depend upon the measurement procedure and may be almost arbitrary [25]. However, we assume that the final states coincide with the eigenstates of an operator A corresponding to a measured quantity. The transition to the general case is straightforward.

First we consider the case of the constant U given by (2), where state vectors $|\varphi_n\rangle$ are eigenstates of the measured operator A and q_n are real constants. These random constants are responsible for different outcomes in a repeated measurement and are the so-called 'hidden variables' of the theory. Since the A-operator and the linear Hamiltonian H_0 commute (in the opposite case experiments do not give required outcomes), state vectors $|\varphi_n\rangle$ are eigenstates of the linear Hamiltonian with energies E_n:

$$i\,d|\varphi_n\rangle/dt = H_0|\varphi_n\rangle = E_n|\varphi_n\rangle.$$

This means that H_0 and U also commute.

Developing the state vector of the system as $|\psi\rangle = \sum_n a_n|\varphi_n\rangle$ we obtain

$$da_n/dt = \gamma a_n(q_n - L) \tag{3}$$

where $L = \sum_n q_n|a_n|^2$. From (3) we get

$$\frac{d}{\gamma dt}\left(\ln\frac{a_i}{a_j}\right) = q_i - q_j \tag{4}$$

and the state $|\varphi_i\rangle$ with the biggest q_i survives. Fortunately, nonlinear equation (3) has the *exact* solution

$$|\psi(t)\rangle = \frac{\exp(-\mathrm{i}(H_0 + U)t)|\psi(t_0)\rangle}{\sqrt{|\psi(t_0)|\exp(-2\mathrm{i}Ut)|\psi(t_0)\rangle}}. \tag{5}$$

In terms of a_i this yields

$$|a_i(t)| = \frac{|\exp(\gamma q_i t)a_i(t_0)|}{\sqrt{\sum_n \exp(2\gamma q_n t)|a_n(t_0)|^2}}. \tag{6}$$

Let us analyse solutions (5), (6). If the quantum system was in the eigenstate $|\varphi_i\rangle$ of the measured operator, then its final state will remain $|\varphi_i\rangle$ with the probability 1. In the general case, the system evolves to the state with the maximum value of q_n. It is worth noting that the evolution goes from a pure state to another pure state. Different outcomes in a repeated measurement are specified by different sets of random variables q_n. Let us calculate the probability p_i of finding the system in a particular state $|\varphi_i\rangle$ after the measurement. Since the state with the biggest q_i survives, p_i equals the probability of finding $q_i > q_n$ for any $n \neq i$, $n = 1, 2, \ldots$ and is given by

$$p_i = \int \ldots \int \omega(q_i)\,\mathrm{d}q_i \prod_{n\neq i} \theta(q_i - q_n)\omega(q_n)\,\mathrm{d}q_n \tag{7}$$

where $\theta(q)$ is the theta-function and $\omega(q_n)$ are the probability distributions of random variables q_n. The measurement theory requires

$$p_i = |\langle\psi(t_0)|\varphi_i\rangle|^2. \tag{8}$$

This is the case for random variables q_n distributed along $(-\infty; 0]$ with probability distributions

$$\omega(q_n) = |\langle\psi(t_0)|\varphi_n\rangle|^2 \exp(|\langle\psi(t_0)|\varphi_n\rangle|^2 q_n). \tag{9}$$

It is easy to check that (7), (9) indeed yields $p_i = |\langle\psi(t_0)|\varphi_i\rangle|^2$.

The distributions (9) for q_n are not unique. Any change in variables in the integral (7), which does not change the integral value and does not change the projection property of (1), provides other distributions with correct outcome probabilities. In addition, the number of random variables may be smaller than the number of system states. If, e.g., ξ is a random variable uniformly distributed along [0, 1] and the q_n are given by

$$q_1 = -\xi$$

$$q_n = -\frac{1}{n}\left(\xi - \sum_k^{n-1} |\langle\psi(t_0)|\varphi_k\rangle|^2\right) - \sum_k^{n-1} \frac{|\langle\psi(t_0)|\varphi_k\rangle|^2}{k} \tag{10}$$

then, operator (2) also models the measurement process. This example demonstrates the minimal number of random variables.

Thus, there are two basic assumptions in the proposed oversimplified scheme. The first is nonlinear evolution (1) and the second is the random operator (2), (9), (10). Then, starting from a state $|\psi(t_0)\rangle$, the quantum system will tend to a state $|\varphi_i\rangle$ with the probability $|\langle\psi(t_0)|\varphi_i\rangle|^2$, as measurement theory requires.

3. The change in random variables

The probability distributions (9) show an undesirable dependence upon the initial state vector. This is the consequence of our choice of the constant U operator. Let us remove this restriction and consider the q_n as functions of the system state. Then, transformations $q_n = u_n/|\langle\psi|\varphi_n\rangle|^2$ eliminate the dependence of (9) upon the initial state vector. Here, the random variables u_n are distributed along $(-\infty; 0]$ with distributions

$$\omega(u_n) = \exp(u_n) \tag{11}$$

and the random U operator possessing property (8) is given by

$$U = \sum_n \mathrm{i}\frac{\gamma u_n}{|\langle\psi|\varphi_n\rangle|^2}|\varphi_n\rangle\langle\varphi_n|. \tag{12}$$

Operator (12) is ill defined for the case where $a_n = \langle\varphi_n|\psi\rangle = 0$. This means that the derivative $\mathrm{d}a_n/\mathrm{d}t$ has a singularity at the point where $a_n = 0$. However, solutions of the motion equation exist and can be found, e.g., from the equation for populations $x_n = |\langle\varphi_n|\psi\rangle|^2$

$$\mathrm{d}x_n/\mathrm{d}t = 2\gamma\left(u_n - x_n\sum_i u_i\right). \tag{13}$$

It is important to note that (13) is the *linear* equation.

Unfortunately, the operator (12) cannot model measurements. It would lead to negative values of state populations. For (12) to be consistent with quantum theory the random variables u_n should be positive. These general requirements are written as

$$\left.\frac{\mathrm{d}x_i}{\mathrm{d}t}\right|_{x_i=1} \leqslant 0 \qquad \text{and} \qquad \left.\frac{\mathrm{d}x_i}{\mathrm{d}t}\right|_{x_i=0} \geqslant 0.$$

The case of a positive U/i in a different scheme has been studied by Dodonov and Mizhari [14]. Using the transformation $q_n \Rightarrow -1/\bar{q}_n$, we avoid the difficulty and find that

$$U = \sum_n -\mathrm{i}\frac{\gamma|\langle\psi|\varphi_n\rangle|^2}{u_n}|\varphi_n\rangle\langle\varphi_n| \tag{14}$$

indeed models measurements. The operator (14) differs from that in the Bohm–Bub theory only by a set of random variables. A number of other operators capable of describing measurements is generated by (14) with the help of suitable transformations.

We note that one of them

$$U = \sum_n \mathrm{i}\gamma\ln\left(\frac{|\langle\psi|\varphi_n\rangle|^2}{|u_n|}\right)|\varphi_n\rangle\langle\varphi_n| \tag{15}$$

results in the nonlinear motion equation which has *exact* 'blow-up' solutions

$$x_i(t) = |u_i|\left(\frac{x_i(t_0)}{|u_i|}\right)^{\exp(2\gamma t)} \Bigg/ \sum_n |u_n|\left(\frac{x_n(t_0)}{|u_n|}\right)^{\exp(2\gamma t)}. \tag{16}$$

Again, $|\varphi_i\rangle$ survives with probability $p_i = |\langle\psi(t_0)|\varphi_i\rangle|^2$.

The most attractive candidates for random variables are phases of eigenstates $|\varphi_i\rangle$. The phases can be represented by random variables χ_i uniformly distributed along $[0, 2\pi]$ (2π is an interval of the phase change). The surprising thing is that $u_n = \ln(\chi_n/2\pi)$ are random variables which have exactly the probability distributions (11). It follows from a simple calculation

$$\omega(u_n) = \frac{1}{2\pi}\int_0^{2\pi} \delta(\ln(\chi_n/2\pi) - u_n)\,\mathrm{d}\chi_n = \exp(u_n).$$

In addition, in the case of an imaginary U operator the phases of a_n are constant. It means that values of $2\chi_n = \ln(\langle\psi|\varphi_n\rangle/\langle\varphi_n|\psi\rangle)$ are also constant. Therefore, we find that the nonlinear operator

$$U = -\sum_n \mathrm{i}\frac{|\langle\psi|\varphi_n\rangle|^2}{\ln|\chi_n/2\pi|}|\varphi_n\rangle\langle\varphi_n| = -\sum_n \mathrm{i}\frac{|\langle\psi|\varphi_n\rangle|^2}{\ln|\mathrm{i}/4\pi\,\ln(\langle\psi|\varphi_n\rangle/\langle\varphi_n|\psi\rangle)|}|\varphi_n\rangle\langle\varphi_n| \tag{17}$$

reproduces all features of operator (14). Thus, (17) can be used to describe measurements. Operator (17) does not contain any additional random variable except ones connected with the state vector of the system. Evolution (1) with (17) pushes the system to achieve the state $|\varphi_i\rangle$ with the probability $p_i = |\langle\psi(t_0)|\varphi_i\rangle|^2$. Averaging is performed over various phases χ_n of eigenstates $|\varphi_n\rangle$.

4. Density matrix dynamics

Finally, let us discuss the evolution of the density operator. There are many distributions of pure states corresponding to the same density operator. In linear quantum mechanics all these distributions evolve in the same way. In general, this is not true for nonlinear dynamics. The density matrix equations corresponding to (1), (14), (17) are not closed form. It was Gisin [17] who showed that such dynamics would make it possible to signal faster than light. Fortunately, the nonlinear evolution with (14), (17) does not give a superliminal line. It follows from the absence of the mean value of the operators (14), (17) due to the fact that $\langle 1/u_n\rangle = -\infty$. Though we can find the dynamics of an initial state vector for each set of parameters u_n, the state vector distribution obtained by averaging over u_n produces no density operator. The interrupted measurement, therefore, will not result in a certain density operator. So, it is impossible to predict the results of measurements which follow the interrupted measurement and to construct a superliminal line in this manner. Naturally, upon completion of the measurement the density matrix satisfies Gisin's criterion [17].

The negative sign of random variables q_n allows us to transform the model with constant operator (2), (9), (10) and eliminate superliminal communications. For this purpose, it is sufficient to introduce an additional 'zero' state corresponding to the absence of the quantum system. The 'zero' state represents the case when a detector gives no outcomes. Then, the system state vector is developed as $|\psi\rangle = \tilde{a}_n|\varphi_n\rangle + a_0|0\rangle$ with the condition $\sum_n |\tilde{a}_n|^2 + |a_0|^2 = 1$. Taking $a_n = N\tilde{a}_n$ and $\mathrm{d}(\ln N)/\mathrm{d}t = \sum_n \gamma q_n |a_n|^2$, we find that $\mathrm{d}\tilde{a}_n/\mathrm{d}t = \gamma q_n \tilde{a}_n$ and $|a_0|^2 = 1 - 1/N^2$. The dynamics of the density operator which does not include the 'zero' state is linear. The dynamics is nonlinear in terms of the a_n corresponding to the conditional probabilities.

5. Problems and conclusions

The most serious problem of the proposed theory is, of course, *ad hoc* nonlinear operators (2), (9), (10), (14), (17). Although they nicely model the measurement procedure, we have no rules with which to choose one of them in a natural way. In addition, it is not clear how such an operator is produced by the interaction of a quantum system with an apparatus. Problems exist with the preferred basis and the strength of the nonlinear terms [8]. In the general case, we need a nonlinear equation which is, to a great accuracy, the Schrödinger equation for a small number of particles and a reduction equation for a huge number of particles. Because the norm of the one-particle state vector is not conserved in a relativistic theory, it appears that we should consider a dynamics that does not conserve the norm of a state vector. Very interesting examples of such evolution have been studied by Ghirardi *et al* [18].

Nevertheless, we believe that the discussed dynamics can be a good approximation to what is really happening in the measurement process. If it is possible to describe a system interaction with a classical detector by means of a closed equation on the normalized state vector, then, equation (1) should be regarded as the only candidate. It opens up a new area of investigation, possesses the attractive projective property and demonstrates exact solutions in several important cases. It allows us to construct models with random variables, which reproduce quantum mechanical results of measurements. Phases of state vectors can be the random variables of the theory. It is interesting, however, to find some physical reason for the proposed evolution.

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